A MULTI-PLAYER EVASION DIFFERENTIAL GAME PROBLEM IN $\mathbb{R}^2$

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DECLARATION

I hereby declare that this work is the product of my research efforts undertaken under the supervision of Dr. Abbas Ja'afaru Badakaya and has not been presented anywhere for the award of a degree or certificate. All sources have been duly acknowledged.

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CERTIFICATION

This is to certify that the research work for this dissertation and the subsequent write-up were carried out by Mustapha Adamu Umar (SPS/13/MMT/00004) under my supervision.

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APPROVAL

This dissertation has been examined and approved for the award of the degree of MASTERS OF SCIENCE IN MATHEMATICS.

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DEDICATION

I dedicate this research work to my Father (Late Alh Adamu Umar), and my Mother, (Hajiya Nafisat Abubakar), may Allah (SWT) reward them abundantly (Amin).

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ABSTRACT

Evasion differential game problem with many pursuers and one evader in $\mathbb{R}^2$ has been studied. The control functions of the players are subject to integral constraints on each coordinates, a sufficient condition for evasion were obtained, evader's strategy are constructed and an illustrative example is given.

CHAPTER ONE

INTRODUCTION

1.1 PREAMBLE

Differential Game is an important area of research which lies in the intersection of Game theory, calculus of variation and optimal control theory. Differential games are group of problems related to the modelling and analysis of conflict in the context of dynamical system more specifically a state variable or variable involve over time according to a differential equation. Calculus of variations is a field in mathematical analysis that deal with maximizing or minimizing functionals. Differential games are related closely with optimal control problem, where there is a single control and single criteria, while differential game theory generalized this to two controls and two criteria, one for each player to be optimized.

A typical differential game problems consists of two players, a pursuer and an evader, with conflicting goals. The motions of the pursuer and the evader are modelled by systems of differential equations, each player attempt to control the state variable of the system into a particular target set by the use of a measurable function called control, through which the player makes his input so as to achieve his goals, the system respond to the input of both players.

Among the early differential game which involves two players in a specified space, such as Homicidal chauffeur game and lion and man game. Pursuit differential game problem involves finding conditions that are not necessarily sufficient for the pursuer to catch the evader, on the other hand evasion differential game problem involves finding conditions that are sufficient for the evader to escape catch from the pursuer. The Homicidal Chauffeur game is a pursuit-evasion game which involves two players, a circular car driver (the pursuer) and a pedestrian (an evader) in a bounded region, the car is faster but less maneuverable while the pedestrian is slower but more maneuverable at any given time, the car driver tries to knock down the pedestrian, who of course does not wish to be smashed down, tries to maneuver his way out. The question is, if their initial positions vary then what is the best strategy that the car driver or the pedestrian can adopt in order to achieve their goals? The solution to this problem can be applied to real-life air combat and to artificial intelligence.

The lion and man game is also a pursuit evasion problem that was studied in the early nineteenth century which involves a lion and a man in a closed and bounded arena, both assumed to have equal maximum speed. What tactics should the lion employ to be sure of his meal? Is it possible for the man to evade capture? Can both of them achieve their goals? If the region is unbounded, can the man survive? This problem caught the attention of many researchers.

Differential game has numerous applications in various field of knowledge some of which include economics and management science, missile guidance, engineering designs, etc. It is a commonly used tool for modelling and analysing economics and management problems which are characterised by both multiperiod and strategic decision making and also in solving conflict between demand (predator) and supply (prey) of a certain product. A recent development of differential game in economics is the stochastic differential game of capitalism which was developed for the purpose of analysing the role of uncertainty in a deterministic game.

1.2 RESEARCH PROBLEM

We consider the differential game problem in which players motions are described by

$$\begin{array}{llll} P_i: & \dot{x}_i(t) & = & a(t)u_i(t), \quad x_i(0) = x_i^0, \quad i = 1, \ldots, m \\ E: & \dot{y}(t) & = & b(t)v(t), \quad y(0) = y^0, \end{array} \tag{1.2.1}$$

where $x_i^0 = (x_{i1}^0, x_{i2}^0)$, $y^0 = (y_1^0, y_2^0)$, $a(t)$, $b(t)$ are scalar measurable functions, with $u(t)$ and $v(t)$, the control functions of the pursuer and evader respectively, such that

$$\int_0^\infty |u_{i1}(t)|^2 dt \le \rho_{i1}^2, \qquad \int_0^\infty |u_{i2}(t)|^2 dt \le \rho_{i2}^2, \tag{1.2.2}$$

$$\int_0^\infty |v_1(t)|^2 dt \le \sigma_1^2, \qquad \int_0^\infty |v_2(t)|^2 dt \le \sigma_2^2 \tag{1.2.3}$$

where ρ_{i1}, ρ_{i2}, σ_1, σ_2, are given positive numbers.

The question is what are the conditions that ensure evader will escape from the catch by the pursuers (i.e., possibility of evasion)?

This problem is a generalization of the problem considered in [6], where $a(t) = b(t) = 1$.

1.3 AIM AND OBJECTIVES

The aim of this research work is to find condition that ensure or guarantees evasion in the differential game problem described by (1.2.1), (1.2.2) and (1.2.3)

The objectives are;

- To construct admissible strategy of the evader.
- To formulate a Theorem that provides sufficient Conditions for Evasion to be possible.
- To give illustrative example.

1.4 SCOPE AND LIMITATIONS

In this research work we consider

(i) $a(t)$ and $b(t)$ described in (1.2.1) to be scalar measurable functions.

(ii) The differential game in the space $\mathbb{R}^2$.

1.5 RESEARCH METHODOLOGY

We use method for solving differential equations from calculus with some inequalities properties to solve our problems.

1.6 DEFINITION OF SOME BASIC TERMS

Here, we give definition of some basic terms;

Definition 1.6.1 *(State variable) State variable of a player is one of the set of variables used to describe the state or position of that player at any given time in the space.*

Definition 1.6.2 *(Constraint) Constraint is a factor that restricts the development or transformation of a player's potential from achieving its goal.*

Definition 1.6.3 *(Control Function) A player's control (or steering device) is a function through which a player uses to make input in order to achieve his goal.*

Definition 1.6.4 *(Admissible Strategy) Admissible Strategy of a player is the strategy that satisfies the constraint imposed on the control function of the player.*

Definition 1.6.5 *(Sigma Algebra) Let X be a nonempty set. A collection Σ of subset of X is called a σ-algebra (sigma-algebra) in X if the following conditions hold :*

i. $\emptyset \in \Sigma$;

ii. If $A \in \Sigma$, then its compliment, A^c, is also in Σ ;

iii. If $\{A_n\}$ is a sequence of sets in Σ, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$, i.e., Σ is closed under countable unions.

Definition 1.6.6 *(Measurable Space) Measurable Space is a set considered together with the sigma algebra on the set.*

Definition 1.6.7 *(Measurable Function) Let* (X, Σ) *be a measurable space and* $E \in \Sigma$. *A function* $f : E \mapsto \mathbb{R}^*$ *is said to be measurable if for each* $\alpha \in \mathbb{R}$, *the set* $\{x \in E : f(x) < \alpha\}$ *belongs to* Σ, *where* $\mathbb{R}^*$ *is the set of extended real numbers.*

CHAPTER TWO

LITERATURE REVIEW

2.1 INTRODUCTION

This chapter contain review of some related works which explore what is obtainable and the gap in the literature, it also give the window through which, we derived our research problem.

2.2 LITERATURE REVIEW

Differential game due to its importance has been an area of great interest to many applied mathematicians and it emanates as a result of inter-field research activities in game theory and optimal control, thus a number of research have been devoted to this field and fundamental results were published in books and journals for example, see ([1, 4, 14, 20, 27]).
Differential game is an extension of (sequential) game theory to the continuous-time case, which is categorised as group of problems that are related to modelling and analysis of conflict problems in the context of a dynamical system, differential games can also be considered as an extension of optimal control problem, which is considered as differential game involving one player, to a game involving two players. Therefore, different approaches were chosen by different authors to solve pursuit-evasion differential games with one or several pursuers.
Pursuit-evasion differential game is a differential game involving at least two players and it is of great importance due to its numerous application to solving real life problems, this motivated many researchers to study such type of differential game see for example

([5, 7, 8, 9, 10, 12, 13, 18, 24, 26]).
some of the researches in relation to pursuit differential game problems can be found in these references ([7, 8, 9, 10, 12, 19]).
However, there are few researches on evasion differential problems compare to that of pursuit differential game problems see these references ([1, 6, 11, 15, 16, 23]).
Azimov, Y. [1] considered evasion differential games problem with one pursuer and one evader the dynamic of the object $z(t)$ is given by the equation

$$\dot{z} = Az - u + v, \quad z(0) = z_0, \tag{2.2.1}$$

where $z \in \mathbb{R}^n$, A is a constant matrix, $u(t)$ and $v(t)$ are the control functions of the pursuer and evader respectively, which are subjected to integral constraint. The subspace M of $\mathbb{R}^n$ is considered as terminal set, they solved the evasion problem from any initial position and obtained sufficient condition if $z(t) \notin M$, $t \geq 0$, then evasion is possible.
Satimov, N.Y. and Rikshev,B.B. [25] studied evasion differential game problem described by linear differential equation of the form,

$$\dot{z} = Az - Bu + Cv, \qquad z(0) = z_0, \tag{2.2.2}$$

where $z \in \mathbb{R}^n$, A,B and C are constant matrices, u and v are control parameters of the pursuer and evader respectively, which are subjected to integral constraint, the terminal set M is a subspace of $\mathbb{R}^n$. They solved the problem and obtained sufficient condition which shows the possibility of evasion if $z(t) \notin M, t \geq 0$.
Ibragimov, *et al.* [6] studied evasion differential game with several pursuers and one evader with integral constraints on control functions of players. Assuming that the total resource of the pursuers does not exceed that of the evader, they solve the game by presenting explicit strategy for the evader which guarantees evasion.
Idham, A.A *et al.* [17] studied a simple motion differential game of infinitely many evaders and infinitely many pursuers in Hilbert space l_2. Control functions of the players are subjected to integral constraints, if the position of an evader never coincides with the position of any pursuer, then evasion is possible.

Samatov, B.T. [24] studied pursuit-evasion problem for a model game with simple motions in the space $\mathbb{R}^n$, where the pursuer controls are subject to both integral and geometric constraints while evader controls are only subject to geometric ones. In this problem depending on initial conditions of the players and parametric values participating in controls constraints, they solve pursuit and evasion problems, this work develops and extends the work of Isaacs, petrosyan and others.

Ibragimov, *et al.* [10] studied differential game of one pursuer and one evader with integral constraints on a closed convex subset S of $\mathbb{R}^n$. Evasion and pursuit problems were investigated and a formula for optimal pursuit time was found and optimal strategies of the players were constructed.

Idham, A.A., *et al.* [16] investigated evasion differential game of two evaders and one pursuer, control functions of all players are subjected to integral constraints. They proved that if the total energy of the evaders is greater than or equal to the energy of the pursuer, then evasion is possible.

Ibragimov, *et al.* [15] studied evasion differential game of simple motion involving one pursuer one evader in $\mathbb{R}^2$ the control are subject to geometric constraints, it is prove that if the maximum speed of the pursuer is equal 1, maximium speed of evader is $\alpha > 1$, control set of the evader is a sector S whose radius is > 1, then condition that guarantee the evasion regardless of the location of initial position of players is obtained.

The work of Ibragimov, G.I and Yusra S. [6] is of special interest in this research work, the evasion differential game problem was studied with one evader in the space $\mathbb{R}^2$, with integral constraints on the control functions of pursuers and evader were considered, they proved that evasion is possible from any initial position $(x_1^0, x_2^0, ..., x_m^0, y^0)$ in the game. This work was extend to $\mathbb{R}^3$ [23], and also sufficient conditions for the possibility of evasion was obtained.

CHAPTER THREE

METHODOLOGY

3.1 INTRODUCTION

This chapter consist of the method we follow in doing the research work. The work of Ibragimov, G.I and Yusra S. [6] is significant to our research work, we review [6] and present the work with some elaborations where necessary.

3.2 EVASION DIFFERENTIAL GAME OF MANY PURSUERS AND ONE EVADER

The evasion problem considered in the paper was studied in $\mathbb{R}^2$, with many pursuers and one evader described by the equations

$$\begin{aligned} &P : \dot{x}_i = u_i, \quad x_i(0) = x_i^0, \quad i = 1, ..., m, \\ &E : \dot{y} = v, \quad y(0) = y^0, \end{aligned} \tag{3.2.1}$$

and assume that $x_i^0 \neq y^0$, $x_i^0 = (x_{i1}^0, x_{i2}^0)$, $i = 1, \ldots, m$, $y^0 = (y_1^0, y_2^0)$.
The following are important definitions in the study:

3.2.1 Admissible control of Pursuers

Definition 3.2.1 *A measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \geq 0$ is called an admissible control of the pursuer x_i if*

$$\int_0^\infty |u_{i1}(s)|^2 ds \leq \rho_{i1}^2, \quad \int_0^\infty |u_{i2}(s)|^2 ds \leq \rho_{i2}^2, \tag{3.2.2}$$

where $\rho_{i1}, \rho_{i2}, i = 1, ..., m$ are given positive numbers.

3.2.2 Admissible control of Evader

Definition 3.2.2 *A measurable function $v(t) = (v_1(t), v_2(t))$, $t \geq 0$ is called an admissible control of the evader y if*

$$\int_0^\infty |v_1(s)|^2 ds \leq \sigma_1^2, \quad \int_0^\infty |v_2(s)|^2 ds \leq \sigma_2^2, \tag{3.2.3}$$

where σ_1, σ_2, are given positive numbers.

Definition 3.2.3 *A function of the form*

$$V(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ f(u_1(t-\varepsilon), ..., u_m(t-\varepsilon)), & t > \varepsilon, \end{cases} \tag{3.2.4}$$

is called the strategy of the evader, where ε is a positive number, $f : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^2$ is a continuous function and $u_1(t), ..., u_m(t)$, $t \geq 0$ are admissible controls of the pursuers.

3.2.3 Condition for Evasion to be possible

Definition 3.2.4 *They say that evasion is possible from the initial position $(x_1^0, x_2^0, ..., x_m^0, y^0)$ in the game (3.2.1) - (3.2.4) if there exist a strategy of the evader V such that $x_i(t) \neq y(t)$, $t \geq 0$, $i = 1, ..., m$ for any admissible controls of the pursuers.*

They now state the evasion problem.

Problem 1.

Find condition for all initial positions $(x_1^0, x_2^0, ..., x_m^0, y^0)$ and parameters σ_1, σ_2, ρ_{i1}, ρ_{i2}, $i =$

$1,...,m$ which guarantee evasion in the game (3.2.1)-(3.2.3). This was the main problem studied in the paper, it should be noted that, in the evasion game, the pursuer use arbitrary admissible controls $u_1(t),...,u_m(t)$, $t \geq 0$, and the evader uses a strategy. By Definition 3.2.4 this strategy is constructed based on values $u_1(t-\varepsilon),...,u_m(t-\varepsilon)$.

After the proof of the main result (Theorem 3.1), they will give illustrative examples. For the initial positions, which do not satisfy hypotheses of the theorem, they have shown that pursuit can be completed, therefore, at first they have to give a definition for "pursuit can be completed", to this end they need to defined strategies of the pursuers.

Definition 3.2.5 *A Borel measurable function $U_i(v) = (U_{i1}(v), U_{i2}(v))$, $U_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is called a strategy of the pursuer x_i if for any control of the evader $v(t)$, $t \geq 0$ and the inequalities*

$$\int_0^{\infty} |U_{i1}(v(s))|^2 ds \leq \rho_{i1}^2, \quad \int_0^{\infty} |U_{i2}(v(s))|^2 ds \leq \rho_{i2}^2 \tag{3.2.5}$$

hold.

3.2.4 Condition for Pursuit to be possible

Definition 3.2.6 *They say that pursuit can be completed from the initial position $(x_1^0, x_2^0, ..., x_m^0, y^0)$ in the game (3.2.1)-(3.2.3) if their exist strategies of the pursuers U_i, $i = 1,...,m$ such that for any admissible control of the evader the equality $x_i(\tau) = y(\tau)$ holds for some $i \in 1,...,m$ and $\tau \geq 0$.*

Note that in the pursuit game, the pursuers use strategies and the evader uses any admissible control $v(t)$ (see Definition 3.2.6), according to Definition 3.2.5 at current time t the pursuers use $v(t)$ to construct their strategies.

3.2.5 Their Main Result

The main results of [6] are presented as follows:

Theorem 3.2.7 *Assume the following conditions hold;*

(1) There exist a subset I of the set $\{1,...,m\}$ *such that*

$$\sum_{i\varepsilon I}\rho_{i1}^{2}\leq\sigma_{1}^{2},\quad \sum_{i\varepsilon J}\rho_{i2}^{2}\leq\sigma_{2}^{2},\quad J=\{1,...,m\}\diagdown I. \tag{3.2.6}$$

(2) $y_1^0\notin[x_1',x_1'']$, $\quad y_2^0\notin[x_2',x_2'']$ *where*

$$x_1'=\min_{i\varepsilon I}x_{i1}^0,\quad x_1''=\max_{i\varepsilon I}x_{i1}^0,\quad x_2'=\min_{i\varepsilon J}x_{i2}^0,\quad x_2''=\max_{i\varepsilon J}x_{i2}^0. \tag{3.2.7}$$

then evasion is possible in the game (3.2.1) - (3.2.3).

3.2.6 Proof of the Theorem

Proof

Let $\rho=(\sum_{i=1}^{m}\sum_{j=1}^{2}\rho_{ij}^{2})^{\frac{1}{2}}$, and let $\varepsilon,\varepsilon<\frac{1}{4\rho^2}\min\{(y_1^0-x_1'')^2,(y_2^0-x_2'')^2\}$, be a positive number. The strategy of the evader is constructed as follows:

$$v_1(t)=\frac{y_1^0-x_{k1}^0}{|y_1^0-x_{k1}^0|}\begin{cases}0, & 0\leq t\leq\varepsilon,\\ (\sum_{i\in I}u_{i1}^2(t-\varepsilon))^{\frac{1}{2}}, & t>\varepsilon,\end{cases} \tag{3.2.8}$$

$$v_2(t)=\frac{y_2^0-x_{l2}^0}{|y_2^0-x_{l2}^0|}\begin{cases}0, & 0\leq t\leq\varepsilon,\\ (\sum_{i\in J}u_{i2}^2(t-\varepsilon))^{\frac{1}{2}}, & t>\varepsilon,\end{cases} \tag{3.2.9}$$

where $u_1(t),...,u_m(t),t\geq0$ are any admissible controls of the pursuers, and $k\in I$, $l\in J$ are arbitrary number, the pursuer use any admissible controls and the evader uses the strategy (3.3.3) - (3.3.4). They have prove that evasion is possible in the game. They examine the first case $y_1^0>x_1''$, $\quad y_2^0>x_2''$. Other cases
$y_1^0>x_1''$, $\quad y_2^0<x_2'$; $\quad y_1^0<x_1'$, $\quad y_2^0>x_2''$; $\quad y_1^0<x_1'$, $\quad y_2^0<x_2'$; can also be considered.
For Case 1;
Suppose that $y_1^0>x_1''$, and $y_2^0>x_2''$ that $y_1^0>x_{k1}''$, $y_2^0>x_{l2}''$ and so.

$$\frac{y_1^0-x_{k1}^0}{|y_1^0-x_{k1}^0|}=1,\quad \frac{y_2^0-x_{l2}^0}{|y_2^0-x_{l2}^0|}=1. \tag{3.2.10}$$

Then (3.3.3) and (3.3.4) take the following form

$$v_1(t) = \begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in I} u_{i1}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{3.2.11}$$

$$v_2(t) = \begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in J} u_{i2}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{3.2.12}$$

The admissibility of the strategy in (3.3.3) can be seen from the following

$$\begin{aligned} \int_0^\infty |v_1(s)|^2 ds &= \int_0^\varepsilon |v_1(s)|^2 ds + \int_\varepsilon^\infty |v_1(s)|^2 ds \\ &= \int_\varepsilon^\infty |v_1(s)|^2 ds \\ &= \int_\varepsilon^\infty |(\sum_{i\in I} u_{i1}^2(s-\varepsilon))^{\frac{1}{2}}|^2 ds \end{aligned}$$

$$\begin{aligned} \int_0^\infty |v_1(s)|^2 ds &= \int_0^\infty |\sum_{i\in I} u_{i1}^2(s)| ds \\ &= \int_0^\infty \sum_{i\in I} |u_{i1}^2(s)| ds = \sum_{i\in I} \int_0^\infty |u_{i1}^2(s)| ds \\ &= \sum_{i\in I} \rho_{i1}^2 \le \sigma_1^2, \end{aligned}$$

The admissible control of (3.3.4) is similar.

They show that evasion is possible on the time interval $[0,\varepsilon]$, that is $y_1(t) - x_{i1}(t) > 0$ and $y_2(t) - x_{i2}(t) > 0$.

Observe that

$$\begin{aligned} \int_0^t |u_{i1}(s)| ds &\le \left(\int_0^t 1^2 ds\right)^{\frac{1}{2}} \left(\int_0^t |u_{i1}(s)|^2 ds\right)^{\frac{1}{2}} \\ &\le \rho\sqrt{t}, \end{aligned} \tag{3.2.13}$$

Let $t \in [0, \varepsilon]$, then for any $i \in I$ we have

$$\begin{aligned} y_1(t) - x_{i1}(t) &= y_1^0 + \int_0^t v_1(s)ds - x_{i1}^0 - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' - \int_0^t |u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \rho\sqrt{t} \\ &\geq y_1^0 - x_1'' - \rho\sqrt{\varepsilon} \\ &\geq \frac{1}{2}(y_1^0 - x_1'') > 0, \end{aligned} \tag{3.2.14}$$

this is by the choice of ε, that is $\rho\sqrt{\varepsilon} < \frac{1}{2}(y_1^0 - x_1'')$.

They now show that evasion is possible on (ε, ∞).

Observed that

$$\begin{aligned} \int_{t-\varepsilon}^t |u_{i1}(s)|ds &\leq \left(\int_{t-\varepsilon}^t 1^2 ds\right)^{\frac{1}{2}} \left(\int_{t-\varepsilon}^t |u_{i1}(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq \sqrt{\varepsilon}\rho, \end{aligned} \tag{3.2.15}$$

If $t \in (\varepsilon, \infty)$ then according to (3.3.6) for any $i \in I$ and using (3.3.10) we have

$$\begin{aligned} y_1(t) - x_{i1}(t) &= y_1^0 + \int_\varepsilon^t v_1(s)ds - x_{i1}^0 - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' + \int_\varepsilon^t \sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} \sqrt{\sum_{i\in I} u_{i1}^2(s)}ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \int_{t-\varepsilon}^t |u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \sqrt{\varepsilon}\rho \\ &\geq \frac{1}{2}(y_1^0 - x_1'') > 0, \end{aligned} \tag{3.2.16}$$

Thus, evasion from the pursuers x_i, $i \in I$ is possible.

Similarly using the fact (3.3.8) holds, then for any $i \in J$ we have

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_0^t v_2(s)ds - x_{i2}^0 - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' - \int_0^t |u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \rho\sqrt{t} \\
&\geq y_2^0 - x_2'' - \sqrt{\varepsilon}\rho \\
&\geq \frac{1}{2}(y_2^0 - x_2'') > 0,
\end{aligned} \tag{3.2.17}
$$

this is by the choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{1}{2}(y_2^0 - x_2'')$.

They now show that evasion is possible on (ε, ∞), if $t \in (\varepsilon, \infty)$ then using (3.3.10) for any $i \in J$ we have:

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_\varepsilon^t v_2(s)ds - x_{i2}^0 - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_\varepsilon^t \sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_0^{t-\varepsilon} \sqrt{\sum_{i\in J} u_{i2}^2(s)}ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \int_{t-\varepsilon}^t |u_{i2}(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &\geq y_2^0 - x_2'' - \sqrt{\varepsilon}\rho \\
&\geq \frac{1}{2}(y_2^0 - x_2'') > 0,
\end{aligned} \tag{3.2.18}
$$

For case 2;

Suppose $y_1^0 > x_1''$, $\quad y_2^0 < x_2'$ which we have $y_1^0 > x_{k1}^0$, $\quad y_2^0 < x_{l2}^0$ for some $k \in I$ and $l \in J$ respectively.

Therefore, $\quad y_1^0 - x_{k1}^0 > 0$, $\quad x_{l2}^0 - y_2^0 > 0$, for which we have

$$
\frac{y_1^0 - x_{k1}^0}{|y_1^0 - x_{k1}^0|} = 1, \quad \frac{x_{l2}^0 - y_2^0}{|x_{l2}^0 - y_2^0|} = 1, \tag{3.2.19}
$$

from (3.3.6) and (3.3.7).

They show evasion is possible in the time interval $[0, \varepsilon]$, that is $y_1(t) - x_{i1}(t) > 0$, $\quad x_{i2}(t) - y_2(t) > 0$, using (3.3.8).

Let $t \in [0,\varepsilon]$, then for any $i \in I$ we have

$$\begin{aligned} y_1(t) - x_{i1}(t) &= y_1^0 + \int_0^t v_1(s)ds - x_{i1}^0 - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' - \int_0^t |u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \rho\sqrt{t} \\ &\geq y_1^0 - x_1'' - \rho\sqrt{\varepsilon} \\ &\geq \frac{3}{4}(y_1^0 - x_1'') > 0, \end{aligned} \tag{3.2.20}$$

this is by the choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{3}{4}(y_1^0 - x_1'')$,

They now show that evasion is possible on (ε,∞), if $t \in (\varepsilon,\infty)$ then using (3.3.10) for any $i \in I$ we have:

$$\begin{aligned} y_1(t) - x_{i1}(t) &= y_1^0 + \int_\varepsilon^t v_1(s)ds - x_{i1}^0 - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' + \int_\varepsilon^t \sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds - \int_0^t u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} \sqrt{\sum_{i\in I} u_{i1}^2(s)}ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \int_{t-\varepsilon}^t |u_{i1}(s)|ds \end{aligned}$$

$$\begin{aligned} y_1(t) - x_{i1}(t) &\geq y_1^0 - x_1'' - \sqrt{\varepsilon}\rho \\ &\geq \frac{3}{4}(y_1^0 - x_1'') > 0, \end{aligned} \tag{3.2.21}$$

Next they show that evasion is possible in the time interval $[0,\varepsilon]$. Let $t \in [0,\varepsilon]$, using the fact (3.3.8) then for any $i \in J$ we have

$$\begin{aligned} x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t u_{i2}(s)ds - y_2^0 - \int_0^t v_2(s)ds \\ &\geq x_2' - y_2^0 + \int_0^t |u_{i2}(s)|ds \\ &\geq x_2' - y_2^0 + \rho\sqrt{t} \\ &\geq x_2' - y_2^0 + \sqrt{\varepsilon}\rho \\ &\geq \frac{3}{4}(x_2' - y_2^0) > 0, \end{aligned} \tag{3.2.22}$$

this is by the choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{3}{4}(x_2' - y_2^0)$,
they now show that evasion is possible on (ε,∞), if $t \in (\varepsilon,\infty)$ then using (3.3.10) for any $i \in J$ we have:

$$\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t u_{i2}(s)ds - y_2^0 - \int_\varepsilon^t v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t u_{i2}(s)ds - \int_\varepsilon^t \sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds \\
&\geq x_2' - y_2^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i2}(s)|ds - \int_0^{t-\varepsilon} \sqrt{\sum_{i\in J} u_{i2}^2(s)}ds \\
&\geq x_2' - y_2^0 + \int_{t-\varepsilon}^t |u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \sqrt{\varepsilon}\rho \\
&\geq \frac{3}{4}(x_2' - y_2^0) > 0,
\end{aligned} \tag{3.2.23}$$

For case 3;
Suppose that $y_1^0 < x_1'$, $y_2^0 > x_2''$; which we have $y_1^0 < x_{k1}^0$, $y_2^0 > x_{l2}^0$ fore some $k \in I$, $l \in J$ are arbitrary points respectively. Therefore $x_{k1}^0 - y_1^0 > 0$, $y_2^0 - x_{l2}^0 > 0$, also $|x_{k1}^0 - y_1^0| > 0$, $|y_2^0 - x_{l2}^0| > 0$,

$$\frac{x_{k1}^0 - y_1^0}{|x_{k1}^0 - y_1^0|} = 1, \quad \frac{y_2^0 - x_{l2}^0}{|y_2^0 - x_{l2}^0|} = 1, \tag{3.2.24}$$

with the strategy in (3.3.3) and (3.3.4)
To show evasion is possible in the time interval $[0,\varepsilon]$, using the fact (3.3.8). Let $t \in [0,\varepsilon]$, then for any $i \in I$ we have

$$\begin{aligned}
x_{i1}(t) - y_1(t) &= x_{i1}^0 + \int_0^t u_{i1}(s)ds - y_1^0 - \int_0^t v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \rho\sqrt{t} \\
&\geq x_1' - y_1^0 + \rho\sqrt{\varepsilon} \\
&\geq \frac{1}{4}(x_1' - y_1^0) > 0,
\end{aligned} \tag{3.2.25}$$

this is by the choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{1}{4}(x_1' - y_1^0)$.

they now show that evasion is possible on (ε, ∞), if $t \in (\varepsilon, \infty)$ then using (3.3.10) for any $i \in I$ we have:

$$\begin{aligned}
x_{i1}(t) - y_1(t) &= x_{i1}^0 + \int_0^t u_{i1}(s)ds - y_1^0 - \int_\varepsilon^t v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |u_{i1}(s)|ds - \int_\varepsilon^t \sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds \\
&\geq x_1' - y_1^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i1}(s)|ds - \int_0^{t-\varepsilon} \sqrt{\sum_{i\in I} u_{i1}^2(s)}ds \\
&\geq x_1' - y_1^0 + \int_{t-\varepsilon}^t |u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \sqrt{\varepsilon}\rho \\
&\geq \frac{1}{4}(x_1' - y_1^0) > 0,
\end{aligned} \tag{3.2.26}$$

For $y_2(t) - x_{i2}(t)$, they have to show the evasion possibility in the time interval $[0, \varepsilon]$. Let $t \in [0, \varepsilon]$, then using (3.3.8) for any $i \in J$ we have;

$$\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_0^t v_2(s)ds - x_{i2}^0 - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' - \int_0^t |u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \rho\sqrt{t}
\end{aligned} \tag{3.2.27}$$

$$\begin{aligned}
y_2(t) - x_{i2}(t) &\geq y_2^0 - x_2'' - \sqrt{\varepsilon}\rho \\
&\geq \frac{1}{4}(y_2^0 - x_2'') > 0,
\end{aligned}$$

this is by choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{1}{4}(y_2^0 - x_2'')$.

Next they now show that evasion is possible on (ε, ∞), if $t \in (\varepsilon, \infty)$ then using the fact

(3.3.10) for any $i \in J$ we have:

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_\varepsilon^t v_2(s)ds - x_{i2}^0 - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_\varepsilon^t \sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds - \int_0^t u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_0^{t-\varepsilon} \sqrt{\sum_{i\in J} u_{i2}^2(s)}ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \int_{t-\varepsilon}^t |u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \sqrt{\varepsilon}\rho \\
&\geq \frac{1}{4}(y_2^0 - x_2'') > 0,
\end{aligned}
\tag{3.2.28}
$$

For case 4;

Suppose that $y_1^0 < x_1', \quad y_2^0 < x_2'$ which we have $y_1^0 < x_{k1}', \quad y_2^0 < x_{l2}'$ for some $k \in I$, $l \in J$ are arbitrary points of respectively. Therefore $\quad x_{k1}^0 - y_1^0 > 0, \quad x_{l2}^0 - y_2^0 > 0$, it implies $|x_{k1}^0 - y_1^0| > 0, \quad |x_{l2}^0 - y_2^0| > 0$,

$$
\frac{x_{k1}^0 - y_1^0}{|x_{k1}^0 - y_1^0|} = 1, \quad \frac{x_{l2}^0 - y_2^0}{|x_{l2}^0 - y_2^0|} = 1, \tag{3.2.29}
$$

with the strategy of the evader above (3.3.3) and (3.3.4)

To show evasion is possible in the time interval $[0,\varepsilon]$. Let $t \in [0,\varepsilon]$, then for any $i \in I$ we have

$$
\begin{aligned}
x_{i1}(t) - y_1(t) &= x_{i1}^0 + \int_0^t u_{i1}(s)ds - y_1^0 - \int_0^t v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \rho\sqrt{t} \\
&\geq x_1' - y_1^0 + \rho\sqrt{\varepsilon}
\end{aligned}
\tag{3.2.30}
$$

$$
x_{i1}(t) - y_1(t) \geq \frac{3}{2}(x_1' - y_1^0) > 0,
$$

this is by the choice of ε, i.e. $\rho\sqrt{\varepsilon} < \frac{3}{2}(x_1' - y_1^0)$.

they now show that evasion is possible on (ε,∞), if $t \in (\varepsilon,\infty)$ then using (3.3.10) for any

$i \in I$ we have:

$$
\begin{aligned}
x_{i1}(t) - y_1(t) &= x_{i1}^0 + \int_0^t u_{i1}(s)ds - y_1^0 - \int_\varepsilon^t v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |u_{i1}(s)|ds - \int_\varepsilon^t \sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds \\
&\geq x_1' - y_1^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{i1}(s)|ds - \int_0^{t-\varepsilon} \sqrt{\sum_{i\in I} u_{i1}^2(s)}ds \\
&\geq x_1' - y_1^0 + \int_{t-\varepsilon}^t |u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \sqrt{\varepsilon}\rho \\
&\geq \frac{3}{2}(x_1' - y_1^0) > 0,
\end{aligned}
\tag{3.2.31}
$$

Next they show that evasion is possible in the time interval $[0, \varepsilon]$, using (3.3.8). Let $t \in [0, \varepsilon]$, then for any $i \in J$ we have

$$
\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t u_{i2}(s)ds - y_2^0 - \int_0^t v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t |u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \rho\sqrt{t} \\
&\geq x_2' - y_2^0 + \sqrt{\varepsilon}\rho \\
&\geq \frac{3}{2}(x_2' - y_2^0) > 0,
\end{aligned}
\tag{3.2.32}
$$

this is by the choice of ε, that is $\rho\sqrt{\varepsilon} < \frac{3}{2}(x_2' - y_2^0)$.
they now show that evasion is possible on (ε, ∞), if $t \in (\varepsilon, \infty)$ then according to (3.3.10) for any $i \in J$ we have:

$$
\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t u_{i2}(s)ds - y_2^0 - \int_\varepsilon^t v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t u_{i2}(s)ds - \int_\varepsilon^t \sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds
\end{aligned}
\tag{3.2.33}
$$

$$
\begin{aligned}
x_{i2}(t)-y_2(t) &\geq x_2'-y_2^0+\left(\int_0^{t-\varepsilon}+\int_{t-\varepsilon}^{t}\right)|u_{i2}(s)|ds-\int_0^{t-\varepsilon}\sqrt{\sum_{i\in J}u_{i2}^2(s)}ds\\
&\geq x_2'-y_2^0+\int_{t-\varepsilon}^{t}|u_{i2}(s)|ds\\
&\geq x_2'-y_2^0+\sqrt{\varepsilon}\rho\\
&\geq \frac{3}{2}(x_2'-y_2^0)>0,
\end{aligned}
$$

Thus evasion is possible from pursuer x_i. Thus complete the proof of the theorem. ■

Example. Consider differential games problem of two pursuers and one evader described by the equations

$$
\begin{aligned}
&P:\dot{x}_i=u_i,\quad x_i(0)=x_i^0,\quad \int_0^{\infty}u_{ij}^2(s)ds\leq\rho_{ij}^2\\
&E:\dot{y}=v,\quad y(0)=y^0,\quad \int_0^{\infty}v_j^2(s)ds\leq\sigma_j^2,\quad i,j=1,2.
\end{aligned}
\tag{3.2.34}
$$

(A) In the first differential game problem, $\rho_{11}^2=4,\rho_{12}^2=1,\rho_{21}^2=1,\rho_{22}^2=4,\sigma_1^2=\sigma_2^2=1$, and hence the sets $I=\{2\},J=\{1\}$ satisfy the first hypothesis of the theorem. Initial positions of the players x_1^0,x_2^0, and y^0, where $x_1^0\neq y^0$, $x_2^0\neq y^0$, are assumed to be any points in the plane. It is not difficult to verify that second hypothesis of the theorem holds, if $x_{21}^0\neq y_1^0$, $x_{12}^0\neq y_2^0$ since $x_1'=x_1''=x_{21}^0$, $x_2'=x_2''=x_{12}^0$. Therefore, for this case conclusion of the theorem is true and hence from such initial positions evasion is possible. Note that the meaning of the condition $x_{21}^0\neq y_1^0$, $x_{12}^0\neq y_2^0$ is that the initial position of the evader y^0 doesn't lie on the horizontal and vertical lines passing through the points x_1^0 and x_2^0, respectively.

They now construct the strategy for the evader, which guarantees the evasion. Let $u_1(t),u_2(t)$, $t\geq 0$ be any admissible controls of the pursuers. According to (3.3.3) and (3.3.4) the strategy of the evader takes the following form:

$$
v_1(t)=\begin{cases}0, & 0\leq t\leq\varepsilon,\\ \xi|u_{21}(t-\varepsilon)|, & t>\varepsilon,\end{cases}\qquad v_2(t)=\begin{cases}0, & 0\leq t\leq\varepsilon,\\ \eta|u_{12}(t-\varepsilon)|, & t>\varepsilon,\end{cases}
\tag{3.2.35}
$$

where

$$\xi = \frac{y_1^0 - x_{21}^0}{|y_1^0 - x_{21}^0|}, \qquad \eta = \frac{y_2^0 - x_{12}^0}{|y_2^0 - x_{12}^0|}. \tag{3.2.36}$$

For this example, $\rho = (\rho_{11}^2 + \rho_{12}^2 + \rho_{21}^2 + \rho_{22}^2)^{\frac{1}{2}}$ and ε satisfies the following conditions:

$$\varepsilon < \frac{1}{4\rho^2} min\Big\{(y_1^0 - x_{21}^0)^2, (y_2^0 - x_{l2}^0)^2\Big\}. \tag{3.2.37}$$

According to the theorem if $x_{21}^0 \neq y_1^0$, $x_{12}^0 \neq y_2^0$, then the strategy of the evader (3.3.30) guarantees the evasion.

Now by considering the case: $y_1^0 > x_{21}^0$, $y_2^0 > x_{12}^0$, they show evasion on the time interval $[0, \varepsilon]$. Let $t \in [0, \varepsilon]$ then for any $i = 2$, $y_1^0 = 3, x_{21}^0 = 2$, we have

$$\begin{aligned} y_1(t) - x_{21}(t) &= y_1^0 + \int_0^t v_1(s)ds - x_{21}^0 - \int_0^t u_{21}(s)ds \\ &\geq 3 - 2 - \int_0^t |u_{21}(s)|ds \\ &\geq 1 - \rho\sqrt{t} \\ &\geq 1 - \rho\sqrt{\varepsilon} \geq 1 - \rho\frac{1}{2\rho} \\ &\geq 1 - \frac{1}{2} = \frac{1}{2} > 0, \end{aligned}$$

Now in the next interval (ε, ∞), if $t \in (\varepsilon, \infty)$ then using the strategy (3.3.30) for any $i = 2$

$$\begin{aligned} y_1(t) - x_{21}(t) &= y_1^0 + \int_\varepsilon^t v_1(s)ds - x_{21}^0 - \int_0^t u_{21}(s)ds \\ &\geq 3 - 2 + \int_\varepsilon^t |u_{21}(s-\varepsilon)|ds - \int_0^t |u_{21}(s)|ds \\ &\geq 1 + \int_0^{t-\varepsilon} |u_{21}(s)|ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|u_{21}(s)|ds \\ &\geq 1 - \int_{t-\varepsilon}^t |u_{21}(s)|ds \\ &\geq 1 - \rho\sqrt{\varepsilon} \geq 1 - \rho\frac{1}{2\rho} \\ &\geq 1 - \frac{1}{2} = \frac{1}{2} > 0, \end{aligned}$$

Thus, evasion from pursuer x_2, $i = 2$ is possible.

Similarly, $y_2^0 = 3, x_{12}^0 = 1$, that is on the second component. They show evasion is possible in the interval $[0,\varepsilon]$. Let $t\in[0,\varepsilon]$ for any $i=1$, we have

$$\begin{aligned}
y_2(t)-x_{12}(t) &= y_2^0+\int_0^t v_2(s)ds - x_{12}^0 - \int_0^t u_{12}(s)ds\\
&\geq 3-1-\int_0^t |u_{12}(s)|ds\\
&\geq 2-\rho\sqrt{t}\\
&\geq 2-\rho\sqrt{\varepsilon}\geq 2-\rho\frac{1}{2\rho}\\
&\geq 2-\frac{1}{2}=\frac{3}{2}>0,
\end{aligned}$$

We now show that evasion is possible on (ε,∞), if $t\in(\varepsilon,\infty)$ then using (3.3.30) for any $i=1$ we have:

$$\begin{aligned}
y_2(t)-x_{12}(t) &= y_2^0+\int_\varepsilon^t v_2(s)ds - x_{12}^0 - \int_0^t u_{12}(s)ds\\
&\geq 3-1+\int_\varepsilon^t |u_{12}(s-\varepsilon)|ds-\int_0^t |u_{12}|(s)ds\\
&\geq 2+\int_0^{t-\varepsilon} |u_{12}(s)|ds-\left(\int_0^{t-\varepsilon}+\int_{t-\varepsilon}^t\right)|u_{i2}(s)|ds\\
&\geq 2-\int_{t-\varepsilon}^t |u_{12}(s)|ds\\
&\geq 2-\sqrt{\varepsilon}\rho\geq 2-\frac{1}{2\rho}\rho\\
&\geq 2-\frac{1}{2}=\frac{3}{2}>0,
\end{aligned}$$

Thus evasion is possible for $i=1$.

However, the theorem gives no information if either $x_{21}^0=y_1^0$, or $x_{12}^0=y_2^0$ since the second hypothesis of the theorem is not satisfied for these cases. They will show that in each of these cases pursuit can be completed in the game (3.3.29) (see Definition 3.2.5 and 3.2.6). In the case $x_{21}^0=y_1^0$, define the strategies of the pursuers as follows:

$$\begin{aligned}
u_{11}(t)=0,\quad u_{12}(t)=0,\quad t\geq 0,\\
u_{21}(t)=v_1(t),
\end{aligned}\tag{3.2.38}$$

$$u_{22}(t)=\begin{cases}\frac{1}{\theta}(y_2^0-x_{22}^0)+v_2(t), & 0\le t\le\theta,\\ 0, & t>\theta,\end{cases} \tag{3.2.39}$$

where $\theta=|y_2^0-x_{22}^0|^2$.

In the case $x_{12}^0=y_2^0$, define the strategies of the pursuers as follows:

$$\begin{aligned}
&u_{21}(t)=0,\quad u_{22}(t)=0,\quad t\ge 0,\\
&u_{11}(t)=\begin{cases}\frac{1}{\theta_1}(y_1^0-x_{11}^0)+v_1(t), & 0\le t\le\theta_1,\\ 0, & t>\theta_1,\end{cases}\\
&u_{12}(t)=v_2(t),
\end{aligned} \tag{3.2.40}$$

where $\theta_1=|y_1^0-x_{11}^0|^2$.

They consider the case $x_{21}^0=y_1^0$. The case $x_{12}^0=y_2^0$ can be analyzed in a similar fashion. Admissibility of the strategy (3.2.38) and (3.2.39) follows from the following relations:

$$\begin{aligned}
\int_0^\infty |u_{22}(t)|^2 &= \int_0^\theta |u_{22}(t)|^2dt+\int_\theta^\infty |u_{22}(t)|^2dt\\
&=\int_0^\theta\left(\frac{1}{\theta}(y_2^0-x_{22}^0)+v_2(t)\right)^2dt\\
&=\int_0^\theta\left(\frac{1}{\theta^2}|y_2^0-x_{22}^0|^2+\frac{2}{\theta}(y_2^0-x_{22}^0)v_2(t)+|v_2(t)|^2\right)dt\\
&=\frac{1}{\theta}|y_2^0-x_{22}^0|^2+\frac{2}{\theta}\int_0^\theta(y_2^0-x_{22}^0)v_2(t)dt+\int_0^\theta|v_2(t)|^2dt\\
&\le 1+\frac{2}{\theta}|y_2^0-x_{22}^0|\int_0^\theta|v_2(t)|dt+\sigma_2^2\\
&\le 2+\frac{2}{\theta}|y_2^0-x_{22}^0|\sqrt{\int_0^\theta 1^2dt}\sqrt{\int_0^\theta|v_2(t)|^2dt}\\
&\le 2+\frac{2\sigma_2^2}{\sqrt{\theta}}|y_2^0-x_{22}^0|=4.
\end{aligned} \tag{3.2.41}$$

Here, they used the Cauchy-Schwartz inequality. Next, we show that $x_2(\theta)=y(\theta)$, that is, pursuit will be completed by the pursuer x_2 at the time θ. We have

$$x_{21}(t)=x_{21}^0+\int_0^t u_{21}(s)ds=y_1^0+\int_0^t v_1(s)ds=y_1(t) \tag{3.2.42}$$

for all $t \geq 0$. In particular, $x_{21}(\theta) = y_1(\theta)$. In addition,

$$\begin{aligned} x_{22}(\theta) &= x_{22}^0 + \int_0^\theta u_{22}(s)ds = x_{22}^0 + \int_0^\theta \left(\frac{1}{\theta}(y_2^0 - x_{22}^0) + v_2(s) \right) ds \\ &= y_2^0 + \int_0^\theta v_2(s)ds = y_2(\theta), \end{aligned} \tag{3.2.43}$$

Here, $x_{22}(\theta) = y_2(\theta)$. Thus $x_2(\theta) = y(\theta)$.

(B) Consider an example of a differential game described by (3.2.42)and (3.2.43), for which the second hypothesis of the theorem is not satisfied. Let $\rho_{11}^2 = 2, \rho_{12}^2 = 1, \rho_{21}^2 = 2, \rho_{22}^2 = 1, \sigma_1^2 = \sigma_2^2 = 2$, $x_1^0 = (0,-1), x_2^0 = (0,1), y^0 = (0,a)$, where $-1 < a < 1$.

Let the strategies of the pursuers be defined by formulas

$$u_{11}(t) = u_{21}(t) = v_1(t), \quad u_{12}(t) = 1, \quad u_{22}(t) = -1, \quad i = 1,2. \tag{3.2.44}$$

They show that pursuit is completed by the time $t = 1$. Indeed, for all $t \geq 0$ we have

$$x_{i1}(t) = x_{i1}^0 + \int_0^t u_{i1}(s)ds = \int_0^t v_1(s)ds = y_1(t), \quad i = 1,2. \tag{3.2.45}$$

What is left is to show that $x_{i2}(\tau) = y_2(\tau)$ for some $i \in 1,2$ and $0 < \tau \leq 1$. It follows from the relations

$x_{12}(0) < y_2(0) < x_{22}$,

$$\begin{aligned} x_{12}(1) &= x_{12}^0 + \int_0^t u_{12}(s)ds = -1 + \int_0^1 1ds = 0, \\ x_{22}(1) &= x_{22}^0 + \int_0^t u_{22}(s)ds = 1 + \int_0^1 (-1)ds = 0, \end{aligned} \tag{3.2.46}$$

that one of the equalities $x_{12}(\tau) = y_2(\tau)$ or $x_{22}(\tau) = y_2(\tau)$ holds at some $\tau, 0 \leq \tau \leq 1$. According to (3.2.45), $x_{i1}(\tau) = y_1(\tau)$ and therefore one of the equalities $x_1(\tau) = y(\tau)$ or $x_2(\tau) = y(\tau)$ holds and hence pursuit is completed at τ. Thus, if the initial position of the evader is in the line segment (x_1^0, x_2^0) then pursuit can be completed by the time $t = 1$.

CHAPTER FOUR

A MULTI-PLAYER EVASION DIFFERENTIAL GAME IN $\mathbb{R}^2$

4.1 INTRODUCTION

This chapter presents the main research problems and their results. The conditions that guarantees evasion were given in form of theorem and its proof, we also present some example.

4.2 STATEMENT OF THE PROBLEM

We consider an evasion differential game of many pursuers and one evader with the equations of motion for the m pursuers and one evader are given by:

$$\begin{aligned} P_i : \dot{x}_i &= a(t)u_i, \quad x_i(0) = x_i^0, \quad i = 1, ..., m, \\ E : \dot{y} &= b(t)v, \quad y(0) = y^0, \end{aligned} \tag{4.2.1}$$

where $x_i^0 = (x_{i1}^0, x_{i2}^0)$, $y^0 = (y_1^0, y_2^0)$, $x_i^0 \neq y^0$, for all i; $a(t)$, $b(t)$ are scalar measurable functions and that there exists a positive number H such that $|a(t)| \leq H \leq |b(t)|$ (the equality holds, if $a(t) = b(t)$). Also we have

$$\int_0^\infty |u_{i1}(t)|^2 dt \leq \rho_{i1}^2, \quad \int_0^\infty |u_{i2}(t)|^2 dt \leq \rho_{i2}^2, \tag{4.2.2}$$

$$\int_0^\infty |v_1(t)|^2 dt \leq \sigma_1^2, \qquad \int_0^\infty |v_2(t)|^2 dt \leq \sigma_2^2, \tag{4.2.3}$$

where ρ_{i1}, ρ_{i2}, σ_1, σ_2, $i = 1, ..., m$ are given positive numbers.
If $a(t)$=$b(t)$=1 then (4.2.1) reduces to the problem studied in [6].

Definition 4.2.1 *A measurable function $u_i(t) = (u_{i1}(t), u_{i2}(t))$, $t \geq 0$ is called an admissible control of the pursuer x_i if the inequalities (4.2.2) are satisfied.*

Definition 4.2.2 *A measurable function $v(t) = (v_1(t), v_2(t))$, $t \geq 0$ is called an admissible control of the evader y if the inequalities (4.2.3) are satisfied*

The solution of the pursuers and evader equation of motion is as follows

$$\begin{aligned} P_i : x_i(t) &= x_i^0 + \int_0^t a(s)u_i(s)ds, \\ E : y(t) &= y^0 + \int_0^t b(s)v(s)ds, \end{aligned} \tag{4.2.4}$$

Definition 4.2.3 *A function of the form*

$$V(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ f(u_1(t-\varepsilon), ..., u_m(t-\varepsilon)), & t > \varepsilon, \end{cases} \tag{4.2.5}$$

is called the strategy of the evader, where ε is a positive number, $f : \mathbb{R}^{2m} \longrightarrow \mathbb{R}^2$ is a continuous function and $u_1(t), ..., u_m(t)$, $t \geq 0$ are admissible controls of the pursuers.

Definition 4.2.4 *If there exist a strategy of the evader V such that $x_i(t) \neq y(t)$, $t \geq 0$ $i = 1, ..., m$ for any admissible controls of the pursuers then we say that evasion is possible from the initial position of the pursuers x_i, $\quad i = 1, ..., m$.*

4.3 CONDITIONS THAT GUARANTEES EVASION

Theorem 4.3.1 *If there exists a subset $I \subset \{1, 2, 3, ..., m\}$, such that*

$$\sum_{i \in I} \rho_{i1}^2 \leq \sigma_1^2, \quad \sum_{i \in J} \rho_{i2}^2 \leq \sigma_2^2, \quad J = \{1, 2, 3, ..., m\} \diagdown I. \tag{4.3.1}$$

and that $y_1^0 \notin [x_1', x_1'']$, $y_2^0 \notin [x_2', x_2'']$, *where*

$$x_1' = \min_{i\in I} x_{i1}^0, \quad x_1'' = \max_{i\in I} x_{i1}^0, \quad x_2' = \min_{i\in J} x_{i2}^0, \quad x_2'' = \max_{i\in J} x_{i2}^0, \tag{4.3.2}$$

then evasion is possible in the game described by (4.2.1)-(4.2.3)

Proof :

Suppose the hypothesis of the theorem holds, we denote $(\sum_{i=1}^{m}\sum_{j=1}^{2}\rho_{ij}^2)^{\frac{1}{2}} = \rho$, and let ε, $\varepsilon < \frac{1}{4\rho^2}\min\{(y_1^0 - x_1'')^2, (y_2^0 - x_2'')^2\}$, be a positive number. We construct a strategy for the evader as follows:

$$v_1(t) = \frac{y_1^0 - x_{k1}^0}{|y_1^0 - x_{k1}^0|}\begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in I} u_{i1}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{4.3.3}$$

$$v_2(t) = \frac{y_2^0 - x_{l2}^0}{|y_2^0 - x_{l2}^0|}\begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in J} u_{i2}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{4.3.4}$$

where $u_1(t), ..., u_m(t), t \ge 0$ are any admissible controls of the pursuers, and $k \in I$, $l \in J$, are arbitrary number, we will prove that evasion is possible if the pursuers uses any admissible controls and the evader uses the strategies.

We now examine evasion problem for the case $y_1^0 > x_1''$, $y_2^0 > x_2''$ other cases such as

$y_1^0 > x_1''$, $y_2^0 < x_2'$; $y_1^0 < x_1'$, $y_2^0 > x_2''$; $y_1^0 < x_1'$, $y_2^0 < x_2'$, can also be considered.

Now for the case $y_1^0 > x_1''$, $y_2^0 > x_2''$ it follows that $y_1^0 - x_1'' > 0$, $y_2^0 - x_2'' > 0$, it shows that $y_1^0 > x_{k1}''$, $y_2^0 > x_{l2}''$ fore some $k \in I$, $l \in J$, are arbitrary number respectively, then $y_1^0 - x_{k1}'' > 0$, $y_2^0 - x_{l2}'' > 0$, and so.

$$\frac{y_1^0 - x_{k1}^0}{|y_1^0 - x_{k1}^0|} = 1, \quad \frac{y_2^0 - x_{l2}^0}{|y_2^0 - x_{l2}^0|} = 1, \tag{4.3.5}$$

Then the evader strategies (4.3.3) and (4.3.4) became

$$v_1(t) = \begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in I} u_{i1}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{4.3.6}$$

$$v_2(t) = \begin{cases} 0, & 0 \le t \le \varepsilon, \\ (\sum_{i\in J} u_{i2}^2(t-\varepsilon))^{\frac{1}{2}}, & t > \varepsilon, \end{cases} \tag{4.3.7}$$

4.3.1 Admissibility of the Evader strategies

The admissibility of the strategy can be seen from the following.

For the first coordinate:

$$\begin{aligned}
\int_0^\infty |v_1(t)|^2 dt &= \int_0^\varepsilon |v_1(t)|^2 dt + \int_\varepsilon^\infty |v_1(t)|^2 dt \\
&= \int_\varepsilon^\infty |v_1(t)|^2 dt \\
&= \int_\varepsilon^\infty |(\sum_{i\in I} u_{i1}^2(t-\varepsilon))^{\frac{1}{2}}|^2 dt \\
&= \int_0^\infty |\sum_{i\in I} u_{i1}^2(s)| ds \\
&= \int_0^\infty \sum_{i\in I} |u_{i1}^2(s)| ds = \sum_{i\in I} \int_0^\infty |u_{i1}^2(s)| ds \\
&= \sum_{i\in I} \rho_{i1}^2 \le \sigma_1^2,
\end{aligned}$$

For the second coordinate we have:

$$\begin{aligned}
\int_0^\infty |v_2(t)|^2 dt &= \int_0^\varepsilon |v_2(t)|^2 dt + \int_\varepsilon^\infty |v_2(t)|^2 dt \\
&= \int_\varepsilon^\infty |v_2(t)|^2 dt \\
&= \int_\varepsilon^\infty |(\sum_{i\in J} u_{i2}^2(t-\varepsilon))^{\frac{1}{2}}|^2 dt \\
&= \int_\varepsilon^\infty |\sum_{i\in J} u_{i2}^2(t-\varepsilon)| dt \\
&= \int_0^\infty |\sum_{i\in J} u_{i2}^2(s)| ds = \sum_{i\in J} \int_0^\infty |u_{i2}^2(s)| ds \\
&= \sum_{i\in J} \rho_{i2}^2 \le \sigma_2^2,
\end{aligned}$$

We observe that by Cauchy schwartz inequality

$$\begin{aligned}\int_0^t H|u_{i1}(s)|ds &\leq \left(\int_0^t H^2 ds\right)^{\frac{1}{2}} \left(\int_0^t |u_{i1}(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq H\sqrt{t}\rho,\end{aligned} \tag{4.3.8}$$

We now show the possibility of evasion in the two intervals $[0,\varepsilon]$ and (ε,∞). Consider the interval $[0,\varepsilon]$, let $t \in [0,\varepsilon]$, for any $i \in I$, using (4.3.8) we have:

$$\begin{aligned}y_1(t) - x_{i1}(t) &= y_1^0 + \int_0^t b(s)v_1(s)ds - x_{i1}^0 - \int_0^t a(s)u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' - \int_0^t |a(s)u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \int_0^t |a(s)||u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - \int_0^t H|u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' - H\sqrt{t}\rho \\ &\geq y_1^0 - x_1'' - H\rho\sqrt{\varepsilon} \geq \frac{1}{2}(y_1^0 - x_1'') > 0,\end{aligned} \tag{4.3.9}$$

this is by choice of ε such that $\rho\sqrt{\varepsilon} < \frac{1}{2H}(y_1^0 - x_1'')$.
We observe that also by Cauchy schwartz inequality

$$\begin{aligned}\int_{t-\varepsilon}^t H|u_{i2}(s)|ds &\leq \left(\int_{t-\varepsilon}^t H^2 ds\right)^{\frac{1}{2}} \left(\int_{t-\varepsilon}^t |u_{i2}(s)|^2 ds\right)^{\frac{1}{2}} \\ &\leq H\sqrt{\varepsilon}\rho,\end{aligned} \tag{4.3.10}$$

We now consider the interval (ε,∞), let $t \in (\varepsilon,\infty)$ then using the strategy (4.3.3) of the evader for any $i \in I$, using (4.3.10) we have:

$$\begin{aligned}y_1(t) - x_{i1}(t) &= y_1^0 + \int_\varepsilon^t b(s)v_1(s)ds - x_{i1}^0 - \int_0^t a(s)u_{i1}(s)ds \\ &\geq y_1^0 - x_1'' + \int_\varepsilon^t b(s)\sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds - \int_0^t |a(s)u_{i1}(s)|ds \\ &\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} |b(s)|\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|a(s)||u_{i1}(s)|ds\end{aligned}$$

$$
\begin{aligned}
y_1(t) - x_{i1}(t) &\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right) H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk - \int_0^{t-\varepsilon} H|u_{i1}(s)|ds - \int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - \int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - H\sqrt{\varepsilon}\rho \geq \frac{1}{2}(y_1^0 - x_1'') > 0,
\end{aligned}
\tag{4.3.11}
$$

Thus, evasion from the pursuers x_i, $i \in I$, is possible.

Similarly with the constructed strategy in (4.3.7).

Now for the time interval $[0,\varepsilon]$. Let $t \in [0,\varepsilon]$ and using (4.3.8.) then for any $i \in J$

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_0^t b(s)v_2(s)ds - x_{i2}^0 - \int_0^t a(s)u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' - \int_0^t |a(s)u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \int_0^t |a(s)||u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \int_0^t H|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - H\sqrt{t}\rho \\
&\geq y_2^0 - x_2'' - H\rho\sqrt{\varepsilon} \geq \frac{1}{2}(y_2^0 - x_2'') > 0,
\end{aligned}
\tag{4.3.12}
$$

this is by the choice of ε, that is $\rho\sqrt{\varepsilon} < \frac{1}{2H}(y_2^0 - x_2'')$.

Now for the interval (ε,∞), let $t \in (\varepsilon,\infty)$ then according to (4.3.7) for any $i \in J$ using (4.3.10) we have:

$$
\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_\varepsilon^t b(s)v_2(s)ds - x_{i2}^0 - \int_0^t a(s)u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_\varepsilon^t b(s)\sqrt{\sum_{i\in J} u_{i1}^2(s-\varepsilon)}ds - \int_0^t |a(s)u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' + \int_0^{t-\varepsilon} |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|a(s)||u_{i2}(s)|ds
\end{aligned}
$$

$$
\begin{aligned}
y_2(t)-x_{i2}(t) &\geq y_2^0-x_2''+\int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J}u_{i2}^2(k)}dk-\left(\int_0^{t-\varepsilon}+\int_{t-\varepsilon}^{t}\right)H|u_{i2}(s)|ds \\
&\geq y_2^0-x_2''+\int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J}u_{i2}^2(k)}dk-\int_0^{t-\varepsilon}H|u_{i1}(s)|ds-\int_{t-\varepsilon}^{t}H|u_{i2}(s)|ds \\
&\geq y_2^0-x_2''-\int_{t-\varepsilon}^{t}H|u_{i2}(s)|ds \\
&\geq y_2^0-x_2''-H\sqrt{\varepsilon}\rho \geq \frac{1}{2}(y_2^0-x_2'')>0,
\end{aligned}
\tag{4.3.13}
$$

Thus, evasion from the pursuers x_i, $i\in J$ is possible.

Secondly, in this case $\quad y_1^0<x_1',\quad y_2^0<x_2',$

it follows from the inequalities that $\quad x_1'-y_1^0>0,\quad x_2'-y_2^0>0,$

$|x_1'-y_1^0|>0,\quad |x_2'-y_2^0|>0,$

clearly

$$
\frac{x_1'-y_1^0}{|x_1'-y_1^0|}=1,\quad \frac{x_2'-y_2^0}{|x_2'-y_2^0|}=1,
\tag{4.3.14}
$$

The constructed strategies of the evader takes the form (4.3.6) and (4.3.7).

Now for the interval $[0,\varepsilon]$, let $t\in[0,\varepsilon]$ then for any $i\in I$ using (4.3.8), we have:

$$
\begin{aligned}
x_{i1}(t)-y_1(t) &= x_{i1}^0+\int_0^t a(s)u_{i1}(s)ds-y_1^0-\int_0^t b(s)v_1(s)ds \\
&\geq x_1'-y_1^0+\int_0^t |a(s)u_{i1}(s)|ds \\
&\geq x_1'-y_1^0+\int_0^t |a(s)||u_{i1}(s)|ds \\
&\geq x_1'-y_1^0+\int_0^t H|u_{i1}|ds \\
&\geq x_1'-y_1^0+H\rho\sqrt{t} \\
&\geq x_1'-y_1^0+H\rho\sqrt{\varepsilon} \\
&\geq \frac{3}{4}(x_1'-y_1^0)>0,
\end{aligned}
\tag{4.3.15}
$$

this is by the choice of ε such that $\rho\sqrt{\varepsilon}<\frac{3}{4H}(x_1'-y_1^0)$.

Now in the interval (ε,∞), let $t\in(\varepsilon,\infty)$ then according to (4.3.6) for any $i\in I$, using

(4.3.10) we have:

$$
\begin{aligned}
x_{i1}(t)-y_1(t) &= x_{i1}^0+\int_0^t a(s)u_{i1}(s)ds - y_1^0 - \int_\varepsilon^t b(s)v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |a(s)u_{i1}(s)|ds - \int_\varepsilon^t |b(s)|v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |a(s)||u_{i1}(s)|ds - \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds \\
&\geq x_1' - y_1^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right) H|u_{i1}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk \\
&\geq x_1' - y_1^0 + \int_0^{t-\varepsilon} H|u_{i1}(s)|ds + \int_{t-\varepsilon}^t H|u_{i1}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk \\
&\geq x_1' - y_1^0 + \int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + H\sqrt{t}\rho \\
&\geq x_1' - y_1^0 + H\rho\sqrt{\varepsilon} \geq \frac{3}{4}(x_1' - y_1^0) > 0,
\end{aligned}
\tag{4.3.16}
$$

Thus, evasion from the pursuers x_i, $i \in I$ is possible.

Now according to (4.3.7), for the interval $[0,\varepsilon]$. Let $t \in [0,\varepsilon]$, using (4.3.8) for any $i \in J$ we have:

$$
\begin{aligned}
x_{i2}(t)-y_2(t) &= x_{i2}^0+\int_0^t a(s)u_{i2}(s)ds - y_2^0 - \int_0^t b(s)v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)||u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \int_0^t H|u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + H\sqrt{t}\rho \\
&\geq x_2' - y_2^0 + H\rho\sqrt{\varepsilon} \geq \frac{3}{4}(x_2' - y_2^0) > 0,
\end{aligned}
\tag{4.3.17}
$$

this is by the choice of $\rho\sqrt{\varepsilon} < \frac{3}{4H}(x_2' - y_2^0)$.

Now for the interval (ε,∞), let $t \in (\varepsilon,\infty)$ then using (4.3.10) for any $i \in J$ we have:

$$\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t a(s)u_{i2}(s)ds - y_2^0 - \int_\varepsilon^t b(s)v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)u_{i2}(s)|ds - \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds \\
&\geq x_2' - y_2^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|a(s)||u_{i2}(s)|ds - \int_0^{t-\varepsilon} |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk \\
&\geq x_2' - y_2^0 + \int_0^{t-\varepsilon} H|u_{i2}(s)|ds + \int_{t-\varepsilon}^t H|u_{i2}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk \\
&\geq x_2' - y_2^0 + \int_{t-\varepsilon}^t H|u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + H\rho\sqrt{\varepsilon} \geq \frac{3}{4}(x_2' - y_2^0) > 0,
\end{aligned} \tag{4.3.18}$$

Thus, evasion from the pursuers x_i, $i \in J$ is possible.

Third case; that is $y_1^0 > x_1''$, $\quad y_2^0 < x_2'$ we have from the inequality that $y_1^0 > x_{k1}^0$, $\quad y_2^0 < x_{l2}^0$, $\quad k \in I$, $\quad l \in J$. Therefore $\quad y_1^0 - x_{k1}^0 > 0$, $\quad x_{l2}^0 - y_2^0 > 0$,

so that

$$\frac{y_1^0 - x_{k1}^0}{|y_1^0 - x_{k1}^0|} = 1, \quad \frac{x_{l2}^0 - y_2^0}{|x_{l2}^0 - y_2^0|} = 1 \tag{4.3.19}$$

and the constructed strategy of the evader take the form in (4.3.6) and (4.3.7).

Now for the time interval $[0,\varepsilon]$, let $t \in [0,\varepsilon]$ then using (4.3.8) for any $i \in I$, we have:

$$\begin{aligned}
y_1(t) - x_{i1}(t) &= y_1^0 + \int_0^t b(s)v_1(s)ds - x_{i1}^0 - \int_0^t a(s)u_{i1}(s)ds \\
&\geq y_1^0 - x_1'' - \int_0^t |a(s)u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - \int_0^t |a(s)||u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - \int_0^t H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - H\sqrt{t}\rho \\
&\geq y_1^0 - x_1'' - H\rho\sqrt{\varepsilon} \geq \frac{1}{4}(y_1^0 - x_1'') > 0,
\end{aligned} \tag{4.3.20}$$

this is by the choice of $\rho\sqrt{\varepsilon} < \frac{1}{4H}(y_1^0 - x_1'')$.

Now for the interval (t,∞), let $t \in (\varepsilon,\infty)$ then according to (4.3.6) for any $i \in I$, using (4.3.10) we have:

$$\begin{aligned}
y_1(t) - x_{i1}(t) &= y_1^0 + \int_\varepsilon^t b(s)v_1(s)ds - x_{i1}^0 - \int_0^t a(s)u_{i1}(s)ds \\
&\geq y_1^0 - x_1'' + \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds - \int_0^t |a(s)u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' + \int_\varepsilon^t H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' + \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk - \int_0^{t-\varepsilon} H|u_{i1}(s)|ds - \int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - \int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\
&\geq y_1^0 - x_1'' - H\rho\sqrt{\varepsilon} \\
&\geq \frac{1}{4}(y_1^0 - x_1'') > 0,
\end{aligned} \tag{4.3.21}$$

Thus, evasion from the pursuers x_i, $i \in I$ is possible.

In the other coordinate $y_2^0 < x_2'$,

we show evasion for the interval $[0,\varepsilon]$, let $t \in [0,\varepsilon]$ then using (4.3.8) for any $i \in J$ we have:

$$\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t a(s)u_{i2}(s)ds - y_2^0 - \int_0^t b(s)v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)||u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + \int_0^t H|u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + H\sqrt{t}\rho \\
&\geq x_2' - y_2^0 + H\rho\sqrt{\varepsilon} \\
&\geq \frac{1}{4}(x_2' - y_2^0) > 0,
\end{aligned} \tag{4.3.22}$$

this is by the choice of $\rho\sqrt{\varepsilon} < \frac{1}{4H}(x_2' - y_2^0)$.

Now for the interval (ε, ∞), let $t \in (\varepsilon, \infty)$ then according to (4.3.7) for any $i \in J$, using (4.3.10) we have:

$$\begin{aligned}
x_{i2}(t) - y_2(t) &= x_{i2}^0 + \int_0^t a(s)u_{i2}(s)ds - y_2^0 - \int_\varepsilon^t b(s)v_2(s)ds \\
&\geq x_2' - y_2^0 + \int_0^t |a(s)u_{i2}(s)|ds - \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds \\
&\geq x_2' - y_1^0 + \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right)|a(s)||u_{i2}(s)|ds - \int_0^{t-\varepsilon} |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk \\
&\geq x_2' - y_1^0 + \int_0^{t-\varepsilon} H|u_{i2}(s)|ds + \int_{t-\varepsilon}^t H|u_{i2}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk \\
&\geq x_2' - y_2^0 + \int_{t-\varepsilon}^t H|u_{i2}(s)|ds \\
&\geq x_2' - y_2^0 + H\sqrt{\varepsilon}\rho \\
&\geq \frac{1}{4}(x_2' - y_2^0) > 0,
\end{aligned} \tag{4.3.23}$$

Thus, evasion from the pursuers x_i, $i \in I$ is possible.

The fourth case $y_1^0 < x_1', \quad y_2^0 > x_2''$ from the inequality it follows that $\quad y_1^0 < x_{k1}^0, \quad y_2^0 > x_{l2}^0$, $k \in I, \quad l \in J$, are arbitrary number respectively. Therefore $\quad x_{k1}^0 - y_1^0 > 0, \quad y_2^0 - x_{l2}^0 > 0$,

$$\frac{x_{k1}^0 - y_1^0}{|x_{k1}^0 - y_1^0|} = 1, \quad \frac{y_2^0 - x_{l2}^0}{|y_2^0 - x_{l2}^0|} = 1, \tag{4.3.24}$$

and the constructed strategy of the evader take the form in (4.3.6) and (4.3.7).

Now for the time interval $[0, \varepsilon]$, let $t \in [0, \varepsilon]$ then using (4.3.8) for any $i \in I$ we have:

$$\begin{aligned}
x_{i1}(t) - y_1(t) &= x_{i1}^0 + \int_0^t a(s)u_{i1}(s)ds - y_1^0 - \int_0^t b(s)v_1(s)ds \\
&\geq x_1' - y_1^0 + \int_0^t |a(s)u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \int_0^t |a(s)||u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + \int_0^t H|u_{i1}(s)|ds \\
&\geq x_1' - y_1^0 + H\sqrt{t}\rho
\end{aligned}$$

$$x_{i1}(t)-y_1(t) \geq x_1'-y_1^0+H\rho\sqrt{\varepsilon} \\ \geq \frac{1}{3}(x_1'-y_1^0) > 0, \tag{4.3.25}$$

this is by the choice of ε, that is $\rho\sqrt{\varepsilon} < \frac{1}{3H}(x_1'-y_1^0)$.

Now for the interval (ε,∞), let $t \in (\varepsilon,\infty)$ then according to (4.3.6) for any $i \in I$, using (4.3.10) we have:

$$\begin{aligned} x_{i1}(t)-y_1(t) &= x_{i1}^0+\int_0^t a(s)u_{i1}(s)ds - y_1^0 - \int_\varepsilon^t b(s)v_1(s)ds \\ &\geq x_1'-y_1^0+\int_0^t |a(s)u_{i1}(s)|ds - \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in I} u_{i1}^2(s-\varepsilon)}ds \\ &\geq x_1'-y_1^0+\left(\int_0^{t-\varepsilon}+\int_{t-\varepsilon}^t\right)H|u_{i1}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk \\ &\geq x_1'-y_1^0+\int_0^{t-\varepsilon} H|u_{i1}(s)|ds+\int_{t-\varepsilon}^t H|u_{i1}(s)|ds - \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in I} u_{i1}^2(k)}dk \\ &\geq x_1'-y_1^0+\int_{t-\varepsilon}^t H|u_{i1}(s)|ds \\ &\geq x_1'-y_1^0+H\rho\sqrt{\varepsilon} \\ &\geq \frac{1}{3}(x_1'-y_1^0) > 0, \end{aligned} \tag{4.3.26}$$

Thus, evasion from the pursuers x_i, $i \in I$ is possible.

Similarly for $y_2(t)-x_{i2}(t)$, using the fact in (4.3.7) from any $i \in J$, we have:

Now for the interval $[0,\varepsilon]$, let $t \in [0,\varepsilon]$. Then using (4.3.8) for any $i \in I$,

$$\begin{aligned} y_2(t)-x_{i2}(t) &= y_2^0+\int_0^t b(s)v_2(s)ds - x_{i1}^0 - \int_0^t a(s)u_{is}(s)ds \\ &\geq y_2^0-x_2''-\int_0^t |a(s)u_{i2}(s)|ds \\ &\geq y_2^0-x_2''-\int_0^t |a(s)||u_{i2}(s)|ds \\ &\geq y_2^0-x_2''-\int_0^t H|u_{i2}(s)|ds \\ &\geq y_2^0-x_2''-H\sqrt{t}\rho \\ &\geq y_2^0-x_2''-H\rho\sqrt{\varepsilon} \\ &\geq \frac{1}{3}(y_2^0-x_2'') > 0, \end{aligned} \tag{4.3.27}$$

this is by the choice of ε, that is $\rho\sqrt{\varepsilon} < \frac{1}{3H}(y_2^0 - x_2'')$.

Now for the interval (ε, ∞), let $t \in (\varepsilon, \infty)$ then according to (4.3.7) for any $i \in J$, using (4.3.10) we have:

$$\begin{aligned}
y_2(t) - x_{i2}(t) &= y_2^0 + \int_\varepsilon^t b(s)v_2(s)ds - x_{i2}^0 - \int_0^t a(s)u_{i2}(s)ds \\
&\geq y_2^0 - x_2'' + \int_\varepsilon^t |b(s)|\sqrt{\sum_{i\in J} u_{i2}^2(s-\varepsilon)}ds - \int_0^t |a(s)u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' + \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t\right) H|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' + \int_0^{t-\varepsilon} H\sqrt{\sum_{i\in J} u_{i2}^2(k)}dk - \int_0^{t-\varepsilon} H|u_{i2}(s)|ds - \int_{t-\varepsilon}^t H|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - \int_{t-\varepsilon}^t H|u_{i2}(s)|ds \\
&\geq y_2^0 - x_2'' - H\sqrt{\varepsilon}\rho \\
&\geq \frac{1}{3}(y_2^0 - x_2'') > 0,
\end{aligned} \tag{4.3.28}$$

Thus, evasion from the pursuers x_i, $i \in J$ is possible. ■

4.4 ILLUSTRATIVE EXAMPLE

We consider differential games of two pursuers and one evader described by the equations

$$\begin{aligned}
&P : \dot{x}_i = sin(t)u_i(t), \quad x_i(0) = x_i^0, \quad \int_0^\infty u_{ij}^2(s)ds \leq \rho_{ij}^2 \\
&E : \dot{y} = (1+e^{-t})v(t), \quad y(0) = y^0, \quad \int_0^\infty v_j^2(s)ds \leq \sigma_j^2, \quad i,j = 1,2.
\end{aligned} \tag{4.4.1}$$

In this differential game, $\rho_{11}^2 = 1, \rho_{12}^2 = 2, \rho_{21}^2 = 2, \rho_{22}^2 = 1, \sigma_1^2 = \sigma_2^2 = 2$, and hence the sets $I = \{2\}, J = \{1\}$ satisfy the first hypothesis of the theorem. Initial positions of the players x_1^0, x_2^0, and y^0, where $x_1^0 \neq y^0$, $x_2^0 \neq y^0$, are assumed to be any points in the plane. It is not difficult to verify that second hypothesis of the theorem holds, if $x_{21}^0 \neq y_1^0$, $x_{12}^0 \neq y_2^0$ since $x_1' = x_1'' = x_{21}^0$, $x_2' = x_2'' = x_{12}^0$. Therefore, for this case conclusion of the theorem is true and hence from such initial positions we can show evasion is possible.

We now construct the strategy for the evader, which guarantees the evasion. Let $u_1(t), u_2(t)$, $t \geq 0$ be any admissible controls of the pursuers. According to (4.3.3) and (4.3.4) the strategy of the evader takes the following form:

$$v_1(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \alpha|u_{21}(t-\varepsilon)|, & t > \varepsilon, \end{cases} \qquad v_2(t) = \begin{cases} 0, & 0 \leq t \leq \varepsilon, \\ \beta|u_{12}(t-\varepsilon)|, & t > \varepsilon, \end{cases} \tag{4.4.2}$$

where

$$\alpha = \frac{y_1^0 - x_{21}^0}{|y_1^0 - x_{21}^0|}, \qquad \beta = \frac{y_2^0 - x_{12}^0}{|y_2^0 - x_{12}^0|}. \tag{4.4.3}$$

For this $\rho = (1+2+2+1)^{\frac{1}{2}}$, $\rho = \sqrt{6}$ and ε satisfies the following conditions:

$$\varepsilon < \frac{1}{4\rho^2} min\Big\{(y_1^0 - x_{21}^0)^2, (y_2^0 - x_{l2}^0)^2\Big\}. \tag{4.4.4}$$

According to the theorem if $x_{21}^0 \neq y_1^0$, $x_{12}^0 \neq y_2^0$, then the strategy of the evader (4.4.2) guarantees the evasion.

Now by considering the case: $y_1^0 > x_{21}^0$, $y_2^0 > x_{12}^0$, we show evasion on the time interval $[0,\varepsilon]$. Let $t \in [0,\varepsilon]$ then for any $i = 2$, $y_1^0 = 6, x_{21}^0 = 3$, with $\rho = \sqrt{6}$, we have

$$\begin{aligned} y_1(t) - x_{21}(t) &= y_1^0 + \int_0^t 1 + e^{-s} v_1(s) ds - x_{21}^0 - \int_0^t sin(s) u_{21}(s) ds \\ &\geq 6 - 3 - \int_0^t |sin(s) u_{21}(s)| ds \\ &\geq 3 - \int_0^t |sin(s)||u_{21}(s)| ds \\ &\geq 3 - \int_0^t 1|u_{21}(s)| ds \\ &\geq 3 - \rho\sqrt{t} \\ &\geq 3 - \sqrt{6}\sqrt{\varepsilon} = 3 - \sqrt{6}\frac{1}{2\sqrt{6}} \\ &\geq 3 - \frac{1}{2} = \frac{5}{2} > 0, \end{aligned}$$

Now in the next interval (ε, ∞), if $t \in (\varepsilon, \infty)$ then using the strategy (4.4.2) for any $i = 2$

$$
\begin{aligned}
y_1(t) - x_{21}(t) &= y_1^0 + \int_\varepsilon^t 1 + e^{-s} v_1(s) ds - x_{21}^0 - \int_0^t sin(s) u_{21}(s) ds \\
&\geq 6 - 3 + \int_\varepsilon^t |1 + e^{-s}||u_{21}(s - \varepsilon)| ds - \int_0^t |sin(s)||u_{21}(s)| ds \\
&\geq 3 + \int_0^{t-\varepsilon} 1|u_{21}(s)| ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^t \right) 1|u_{21}(s)| ds \\
&\geq 3 + \int_0^{t-\varepsilon} 1|u_{21}(s)| ds - \int_0^{t-\varepsilon} 1|u_{21}(s)| ds - \int_{t-\varepsilon}^t 1|u_{21}(s)| ds \\
&\geq 3 - \int_{t-\varepsilon}^t |u_{21}(s)| ds \\
&\geq 3 - \rho\sqrt{\varepsilon} = 3 - \sqrt{6}\frac{1}{2\sqrt{6}} \\
&\geq 3 - \frac{1}{2} = \frac{5}{2} > 0,
\end{aligned}
$$

This is by choice of $\varepsilon < \frac{1}{16\rho^2} min\{4, 9\}$. Thus, evasion from pursuer x_2, $i = 2$ is possible.

Similarly, $y_2^0 = 6, x_{12}^0 = 4$, that is on the second component, now for the interval $[0, \varepsilon]$. Let $t \in [0, \varepsilon]$ for any $i = 1$, we have

$$
\begin{aligned}
y_2(t) - x_{12}(t) &= y_2^0 + \int_0^t 1 + e^{-s} v_2(s) ds - x_{12}^0 - \int_0^t sin(s) u_{12}(s) ds \\
&\geq 6 - 4 - \int_0^t |sin(s) u_{12}(s)| ds \\
&\geq 2 - \int_0^t |sin(s)||u_{12}(s)| ds \\
&\geq 2 - \int_0^t 1|u_{12}(s)| ds \\
&\geq 2 - \rho\sqrt{t} \geq 2 - \sqrt{6}\sqrt{\varepsilon} \\
&\geq 2 - \sqrt{6}\frac{1}{2\sqrt{6}} \\
&\geq 2 - \frac{1}{2} = \frac{3}{2} > 0,
\end{aligned}
$$

We now show that evasion is possible on (ε, ∞), let $t \in (\varepsilon, \infty)$ then using (4.4.2) for any $i = 1$ we have:

$$\begin{aligned}
y_2(t) - x_{12}(t) &= 6 + \int_{\varepsilon}^{t} 1 + e^{-s} v_2(s) ds - 4 - \int_0^t sin(s) u_{12}(s) ds \\
&\geq 6 - 4 + \int_{\varepsilon}^{t} |1 + e^{-s}||u_{12}(s-\varepsilon)| ds - \int_0^t |sin(s)||u_{12}|(s) ds \\
&\geq 2 + \int_0^{t-\varepsilon} 1 |u_{12}(s)| ds - \left(\int_0^{t-\varepsilon} + \int_{t-\varepsilon}^{t} \right) 1 |u_{i2}(s)| ds \\
&\geq 2 + \int_0^{t-\varepsilon} 1 |u_{12}(s)| ds - \int_0^{t-\varepsilon} 1 |u_{i2}(s)| ds - \int_{t-\varepsilon}^{t} 1 |u_{i2}(s)| ds \\
&\geq 2 - \int_{t-\varepsilon}^{t} |u_{12}(s)| ds \\
&\geq 2 - \sqrt{\varepsilon} \rho \geq 2 - \frac{1}{2\sqrt{6}} \sqrt{6} \\
&\geq 2 - \frac{1}{2} = \frac{3}{2} > 0,
\end{aligned}$$

This is by choice of $\varepsilon < \frac{1}{16\rho^2} min\{4, 9\}$. Thus evasion is possible for $i = 1$.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 INTRODUCTION

This chapter summarizes the dissertation, it also contains some observations, conclusion and recommendations are also attached to the chapter.

5.2 SUMMARY

The research work focused on multiplayer evasion differential games which comprises many pursuers and one evader with integral constraints on players control functions, it comprises five chapters and the content of these chapters are summarized below;

The first chapter contains a brief preamble on differential game, statement of research problem, aims and objectives, its scopes and limitations, research methodology and definition of some basic terms.

Chapter two contains literature review on the works by scholars that are related to this research problem. In the third chapter, we present a paper [6] from which we formulated our research problem as published but made some elaborations where necessary.

The major part of the work is in chapter four. Where stated our research problem, which is an extension of the problems reviewed in chapter three. We also present the solution of the research problems, conditions that guarantees the possibility of evasion were stated and proved.

5.3 CONCLUSION

We have studied multiplayer evasion differential games of many pursuers and one evader in the space $\mathbb{R}^2$ with motions of players described by first order differential equation, control functions of the players were subjected to integral constraints, in our research work we have constructed Evasion strategies and shown their admissibility a theorem that provides sufficient conditions for evasion was formulated and proved, more over illustrative example was given, we used some basic calculus and established mathematical inequalities to prove our main result.

5.4 RECOMMENDATIONS

The study examined a multiplayer evasion differential game of many pursuers and one evader in the space $\mathbb{R}^2$, clear understanding of the concept will motivate and encourage future researchers.

- The game can also be considered in another space different from $\mathbb{R}^2$.
- If the hypothesis of the theorem are not satisfied, the constraint can also be changed and also another strategy can be constructed.
- First order differential equation was considered in the paper, second order or finite order differential equation may be considered.

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